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RESEARCH MEMORANDUM

AN APPROXIMATE METHOD FOR ESTIMATING THE INCOMPRESSIBLE LAMINAR BOUNDARY-LAYER CHARACTERISTICS ON A FLAT PLATE IN SLIPPING FLOW

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NATIONAL ADVISORY COMMITTEE
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AN APPROXIMATE METHOD FOR ESTIMATING THE INCOMPRESSIBLE LAMINAR BOUNDARY-LAYER CHARACTERISTICS ON A FLAT PLATE IN SLIPPING FLOW

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SUMMARY

An approximate method is presented for the estimation of the properties of the incompressible laminar boundary layer on a flat plate in the slip-flow region using Kármán's momentum method.

At equivalent stations on the same body at the same Reynolds number, the total thickness and the skin friction of a slipping boundary layer are less than that of the normal boundary layer at the same Reynolds number. However, the difference between the slip and normal boundary layer is small until slip velocities at the wall are encountered which are large in comparison with the free-stream velocity; that is, $u_W \geqq 0.3u_o$. An important effect of slip is that on the displacement thickness of the boundary layer.

The following criterion is presented for determining the importance of slip phenomena: If $\frac{M\lambda}{l} > 0.04$, slip becomes an important factor in describing viscous phenomena.

INTRODUCTION

Recent interest in very high altitude flight has led to an interest in and considerable speculation as to the nature of gas flows when the mean free path of the gas molecules is of the order of magnitude of the boundary-layer thickness on a body and also for which the mean free path of the molecules is of the order of magnitude of the length of the body itself. Tsien (reference 1) has described these two types of flow, the former being called the slip-flow regime and the latter the free-molecule-flow regime.

There has been considerable work done on shear or drag forces in the slip-flow region (see references 2, 3, and 4), but most of this work has been done on bounded flows such as the flow between two concentric rotating cylinders or the flow through long tubes. It is the purpose of this paper to investigate the general nature of the slip flow on a flat plate when the mean free path is of the order of, but less than, the boundary-layer thickness. The analysis is only an approximation, but the properties of a laminar boundary layer in the slip-flow regime and the magnitude of the drag reduction due to slip are evaluated. Insofar as a simple method of evaluation is useful, such an approximate analysis may be justified.

SYMBOLS

C_D	drag coefficient
k	ratio (δ/λ)
l	length of flat plate
L	length of order of mean free path
m	mass of molecule
u	horizontal velocity
v	vertical velocity
x	horizontal coordinate
y	vertical coordinate
δ	boundary-layer thickness
λ	mean free path
μ	viscosity
ν	kinematic viscosity
ρ	density
τ	shear stress

Subscripts:

n	normal flow
o	free-stream values
s	slip flow
w	wall values

ANALYSIS

Kármán's momentum theory for the boundary layer on a flat plate may be expressed (see reference 5) by the formula

$$\tau_w = \frac{d}{dx}\int_0^\delta \rho u(u_o - u)dy \tag{1}$$

This formula may be used if the normal boundary-layer assumptions are valid; that is, that the boundary-layer thickness is small compared with the distance to the leading edge of the plate, that the flow in the boundary layer is almost parallel to the surface, and that the major viscous terms are of the same order of magnitude as the inertia terms. It will be seen as the analysis progresses that these conditions are met, and so a velocity profile for the laminar boundary layer consistent with the boundary conditions in slip flow will be assumed and used to solve equation (1) for the rate of growth of the boundary layer and the value of the surface friction at a given station.

The velocity at the wall in a slip flow from elementary kinetic considerations (see reference 4, pp. 291–299) may be taken as

$$u_w = \lambda\left(\frac{du}{dy}\right)_w \tag{2}$$

so that if a boundary-layer profile of the form

$$\frac{u}{u_o} = A + B\frac{y}{\delta} + C\left(\frac{y}{\delta}\right)^2 \tag{3}$$

is assumed, the following boundary conditions may be taken

$$\left.\begin{aligned} y &= 0 \qquad u = u_w = \lambda\left(\frac{du}{dy}\right)_w \\ y &= \delta \qquad u = u_o \quad \text{and} \quad \frac{du}{dy} = 0 \end{aligned}\right\} \tag{4}$$

The application of these boundary conditions results in

$$\frac{u}{u_o} = \frac{2\lambda}{2\lambda + \delta} + \frac{2y}{2\lambda + \delta} - \frac{y^2}{\delta(2\lambda + \delta)} \tag{5}$$

The friction at the wall is

$$\tau_w = \mu\left(\frac{du}{dy}\right)_w = \frac{2\mu u_o}{2\lambda + \delta} \tag{6}$$

Upon substituting equations (5) and (6) in equation (1) and assuming the flow to be incompressible, the following result is obtained by carrying out the integration

$$\tau_w = \frac{2\mu u_o}{2\lambda + \delta} = \rho u_o^2 \frac{d}{dx}\left[\frac{\frac{2}{3}\lambda\delta^2 + \frac{2}{15}\delta^3}{(2\lambda + \delta)^2}\right] \tag{7}$$

Equation (7) is differentiated to obtain

$$\frac{2\mu u_o}{2\lambda + \delta} = \rho u_o^2\left[\frac{\frac{4}{3}\lambda\delta + \frac{2}{5}\delta^2}{(2\lambda + \delta)^2} - \frac{\frac{4}{3}\lambda\delta^2 + \frac{4}{15}\delta^3}{(2\lambda + \delta)^3}\right]\frac{d\delta}{dx} \tag{8}$$

This equation is now integrated and yields

$$x = \frac{u_o}{\nu}\left[\frac{\delta^2}{10} - \frac{8}{15}\lambda^2 + \frac{16}{15}\frac{\lambda^3}{(2\lambda + \delta)} - \frac{1}{15}\frac{\delta^3}{2\lambda + \delta} + \frac{8}{15}\lambda^2 \log\frac{2\lambda + \delta}{2\lambda}\right] \quad (9)$$

This equation relates the distance from the leading edge of a flat plate and the boundary-layer thickness for a given mean free path. Equation (9) may be simplified by the introduction of the plate length l and the ratio

$$\frac{\delta}{\lambda} = k$$

whereby equation (9) becomes

$$\frac{x}{l} = \frac{u_o l}{\nu}\left(\frac{\lambda}{l}\right)^2\left(\frac{k^2}{10} - \frac{8}{15} + \frac{16}{15}\frac{1}{2 + k} - \frac{1}{15}\frac{k^3}{2 + k} + \frac{8}{15}\log\frac{2 + k}{2}\right) \quad (10)$$

or

$$\frac{x}{l} = R_n\left(\frac{\lambda}{l}\right)^2 f(k) \quad (11)$$

Equation (11) gives the position on a flat plate at which the boundary layer is k times thicker than the mean free path. The value of $f(k)$ is plotted against k in figure 1.

From equation (2) the velocity at the wall at any point is given by

$$u_w = \frac{2u_o}{2 + k} \quad (12)$$

and from equation (6) the friction stress at the wall divided by twice the dynamic pressure is

$$\frac{\tau_w}{\rho u_o^2} = \frac{1}{R_n}\left(\frac{l}{\lambda}\right)\frac{2}{2 + k} \quad (13)$$

The displacement thickness of the boundary layer is found to be

$$\frac{\delta^*}{\delta} = \int_0^1 \left(1 - \frac{u}{u_o}\right) d\left(\frac{y}{\delta}\right) = \frac{k}{6 + 3k} \tag{14}$$

It is readily seen that if the mean free path λ is placed equal to zero in equation (5) the boundary-layer profile becomes

$$\frac{u}{u_o} = 2\frac{y}{\delta} - \frac{y^2}{\delta^2}$$

and the boundary-layer thickness is found, by putting $\lambda = 0$ in equation (9), to be

$$\delta = 5.48\sqrt{\frac{\nu x}{u_o}}$$

which is in general agreement with the Blasius solution.

EXAMPLE

A specific example is now worked out to illustrate for a particular case the difference between a slip and a normal flow. Thus a Reynolds number of 100 was assumed and the ratio of plate length to mean free path was chosen as 25. This might correspond to a 1-foot-chord plate traveling at a Mach number around 2.9 at an altitude of 250,000 feet, since (see appendix A)

$$M = \frac{R_n}{1.37}\frac{\lambda}{l} \tag{15}$$

A lower velocity might have been chosen for a 1-foot body at 250,000 feet, so that the flow would be incompressible, but the resulting lower Reynolds number would have given a larger boundary layer and the slip-flow region would have been confined to a somewhat smaller region near the leading edge so that it would have been more difficult to demonstrate the results of the

slip. Indeed, this fact indicates the desirability of extending the analysis to include the effects of compressibility.

Figure 2 shows the solution for the thickness of the boundary layer along the plate in terms of the dimensionless ratios δ/λ and x/l. The velocity profiles are also plotted at their proper positions. It is seen that there is a slip velocity at the wall over the entire plate.

Figure 3 shows this solution compared with the normal boundary-layer solution at the same Reynolds number. It is seen that the slipping boundary layer is thinner at equivalent stations than the normal boundary layer (taken in this example to be the solution when $\lambda = 0$).

Figure 4 shows a comparison of the displacement thicknesses in terms of the ratio δ^*/l for the two cases. It can be seen that there is a large effect on the displacement thickness due to slip, as might be expected.

Finally, figure 5 shows a comparison of the local skin frictions in the two cases. It may be seen that as the thickness of the boundary layer becomes large with respect to the mean free path, the slip skin friction approaches the normal value. But at the leading edge where, since there is no boundary layer, the flow must be a free-molecule flow, the result of this analysis is compared with the free-molecule stress coefficient given by (see appendix B):

$$\tau_w = \frac{0.366}{M} = \frac{0.366}{2.91} = 0.125$$

It is seen that the present method yields a stress at the leading edge that is just twice the value derived from free molecule considerations. If these skin frictions are integrated over the surface of the plate, the drag coefficient for the slip flow on one surface is found to be

$$C_{d_s} = 0.1312$$

while for the normal flow at the same Reynolds number it is found to be

$$C_{d_n} = 0.1460$$

The figure shows that the effect of slip has little effect on the skin friction over most of the plate.

DISCUSSION

Strictly speaking, the equation for the velocity at the wall

$$u_w = \lambda \left(\frac{du}{dy}\right)_w$$

holds only for the boundary layer when the mean free path is considerably less than the boundary-layer thickness. The error is principally that, in deriving the equation for the velocity at the wall, the momentum brought in to the wall by molecules at an average distance L from the wall is (see reference 4, p. 140)

$$m\left[u_w + L\left(\frac{du}{dy}\right)_w\right]$$

It may be seen from the shape of the velocity profile in figure 6 that this assumption becomes feasible when the boundary-layer thickness is approximately twice the length L or approximately twice the mean free path. It is, therefore, obvious that this type of analysis is only applicable to boundary-layer regions when the mean free path is about one-half or less than one-half the boundary-layer thickness.

From the analysis it is seen that the thicknesses and rate of growth of the slip boundary layer are of the same order of magnitude as those of a normal boundary layer at the same Reynolds number, and hence the boundary-layer assumptions necessary for equation (1) must be equally valid for the slip boundary layer.

In general, the effect of slip is to decrease the drag and the boundary-layer thickness from what would be calculated for a normal boundary layer at the same Reynolds number. The greatest effect of slip is upon the displacement thickness. The growth of the boundary layer and the skin friction at the wall may be very close to the normal values even in the presence of a considerable slip velocity at the wall; that is, $u_w = 0.3u_o$.

It should be noted that further boundary conditions may be imposed by assuming higher powers of y/δ in the equation of the velocity profile. The most obvious condition neglected by this analysis is

$$\left(\frac{\partial^2 u}{\partial y^2}\right)_w = \frac{u_w}{\nu}\left(\frac{\partial u}{\partial x}\right)_w$$

This condition leads to results in somewhat better agreement with the Blasius results for the case of $\lambda = 0$ where $u_w = 0$ so that

$$\left(\frac{\partial^2 u}{\partial y^2}\right)_w \equiv 0$$

but leads to difficulties in the analysis when the mean free path is appreciable and there exists a slip velocity at the wall.

From the example it may be seen that if the boundary layer at the end of the plate has a thickness less than about 20 times the mean free path, it might be expected that the effects of slip would be important. From this fact it is possible to construct, with the aid of equations (11) and (15), an approximate criterion for the importance of slip phenomena. Upon substituting equation (15) into equation (11) and putting $\frac{x}{l} = 1.0$ for the trailing edge, there results

$$1 \approx 1.4\frac{M\lambda}{l}f(k)_{TE}$$

If k is to be less than 20 at the trailing edge, then $f(k)_{TE}$ must be less than 16.5, or roughly

$$\frac{M\lambda}{l} > \frac{1}{25} \tag{16}$$

From this approximate criterion for the importance of slip-flow effects, it may be seen that slip phenomena will be more important at high Mach numbers and the present analysis should be extended to include the effects of compressibility. This criterion agrees well with the slip-flow regime as defined by Tsien. Further, it may be seen that the two fundamental variables most useful to describe gas flows at low densities are Mach number and the ratio of mean free path to body length.

CONCLUSIONS

1. An approximate method of estimating the slip-flow boundary layer on a flat plate has been presented.

2. At equivalent stations the total thickness and the skin friction of a slipping boundary layer are less than that of the normal boundary layer at the same Reynolds number.

3. The difference between the slip and normal boundary layer is small until slip velocities at the wall are encountered which are large in comparison with the free-stream velocity, that is, $u_w \geqq 0.3u_o$.

4. An important effect of slip is that on the displacement thickness of the boundary layer.

5. The following criterion is presented for determining the importance of slip phenomena: If $\frac{M\lambda}{l} > 0.04$, slip becomes an important factor in describing viscous phenomena.

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APPENDIX A

DERIVATION OF EQUATION (15)

Reynolds number is defined as

$$R_n = \frac{\rho u l}{\mu} = \frac{ul}{\lambda}\frac{\rho\lambda}{\mu}$$

and, from the kinetic theory of gases,

$$\mu = 0.499\rho\bar{c}\lambda$$

so that practically

$$R_n = 2\frac{u}{\bar{c}}\frac{l}{\lambda}$$

Since the mean molecular velocity $\bar{c}$ for air is 1.462 times the velocity of sound a, there results

$$R_n = 1.37M\frac{l}{\lambda}$$

or

$$M = \frac{R_n\lambda/l}{1.37}$$

APPENDIX B

DERIVATION OF FREE-MOLECULE FRICTION STRESS COEFFICIENT

The momentum which strikes a unit area in a unit time of a flat plate in a free-molecule flow is $1/4\rho\bar{c}u_o$. If all the molecules are reflected with zero velocity from the surface of the plate, the shear stress at the surface is

$$\tau_w = \frac{1}{4}\rho\bar{c}u_o$$

The stress coefficient is therefore

$$\frac{\tau_w}{\rho u_o^2} = \frac{1}{4}\frac{\bar{c}}{u_o} = \frac{1}{4}\frac{1.462}{M} = \frac{0.366}{M}$$

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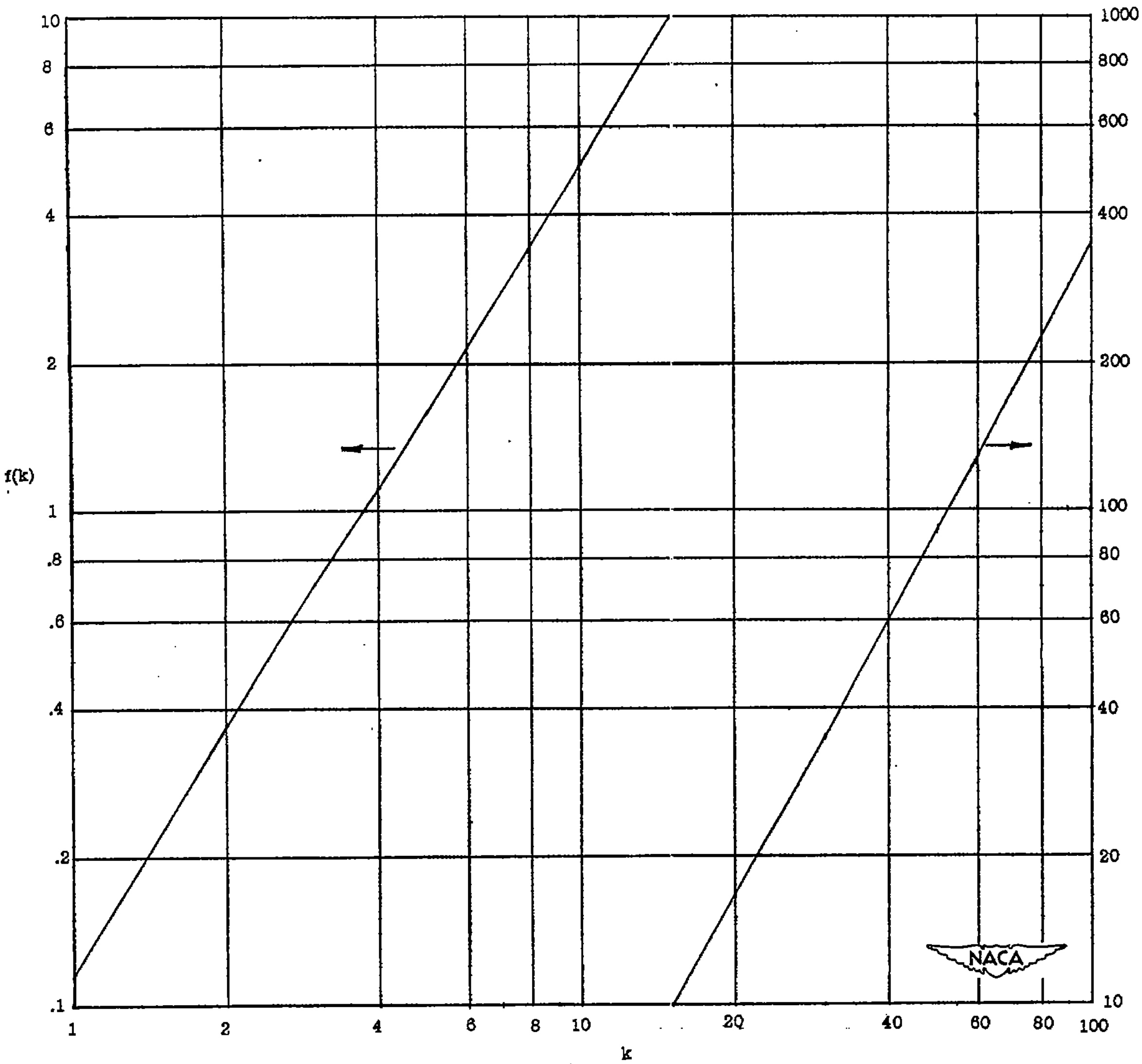

Figure 1.- Function of k from equation (11) plotted against k.

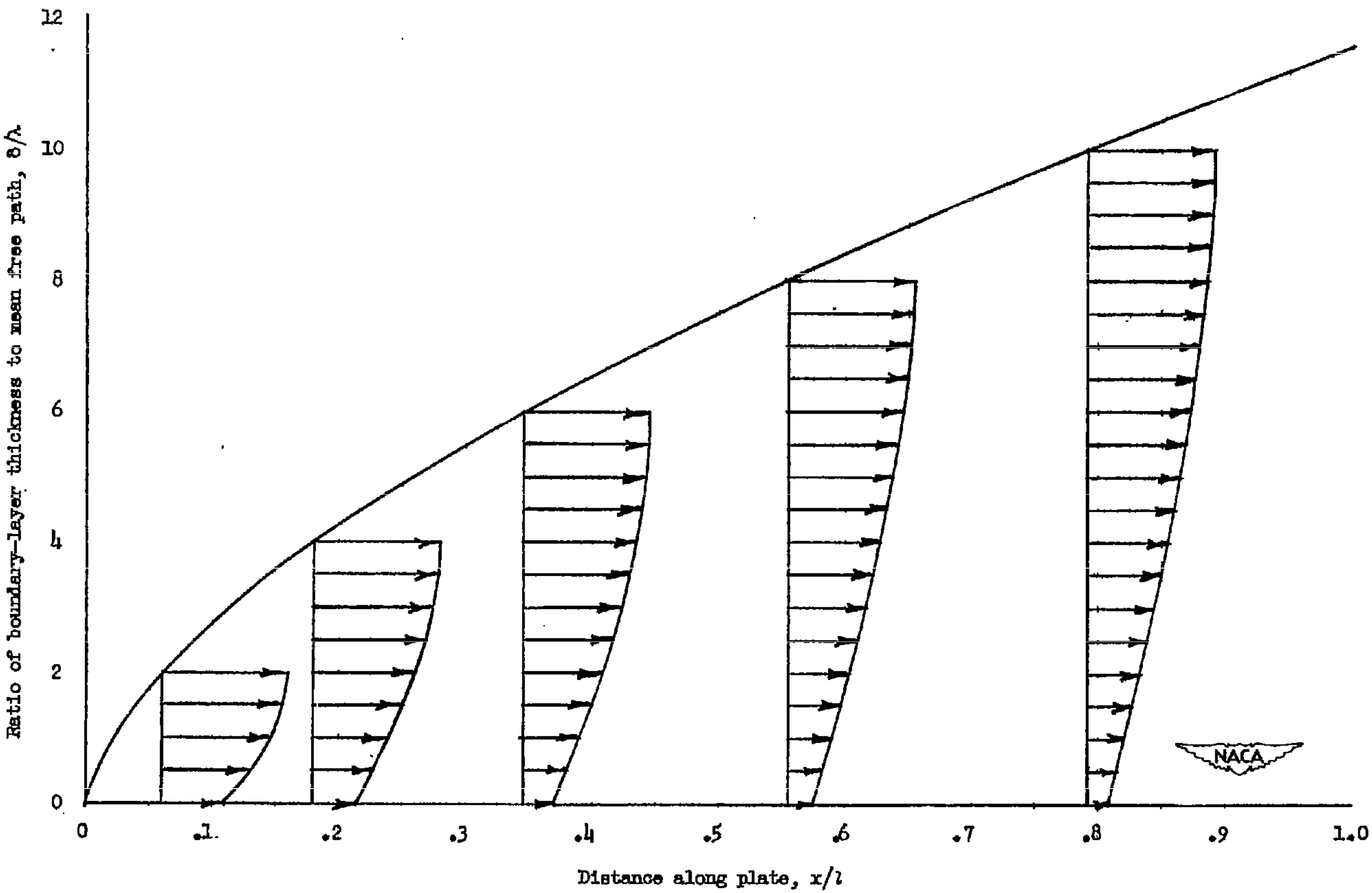

Figure 2.– Boundary layer on a flat plate in slip flow. $R_N = 100$; $\frac{l}{\lambda} = 25$.

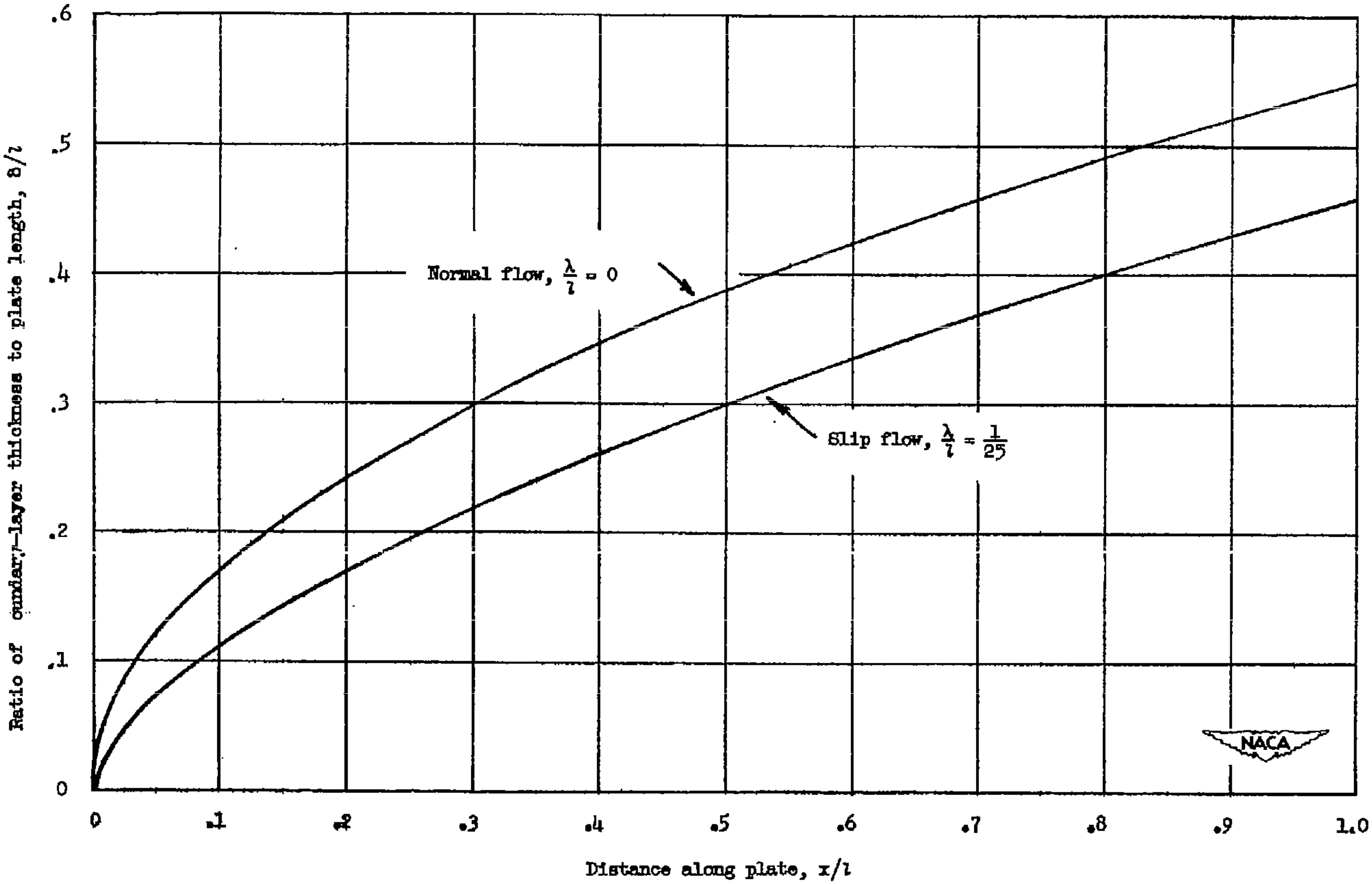

Figure 3.– Comparison of the normal and slip boundary layers on a flat plate. $R_N = 100$.

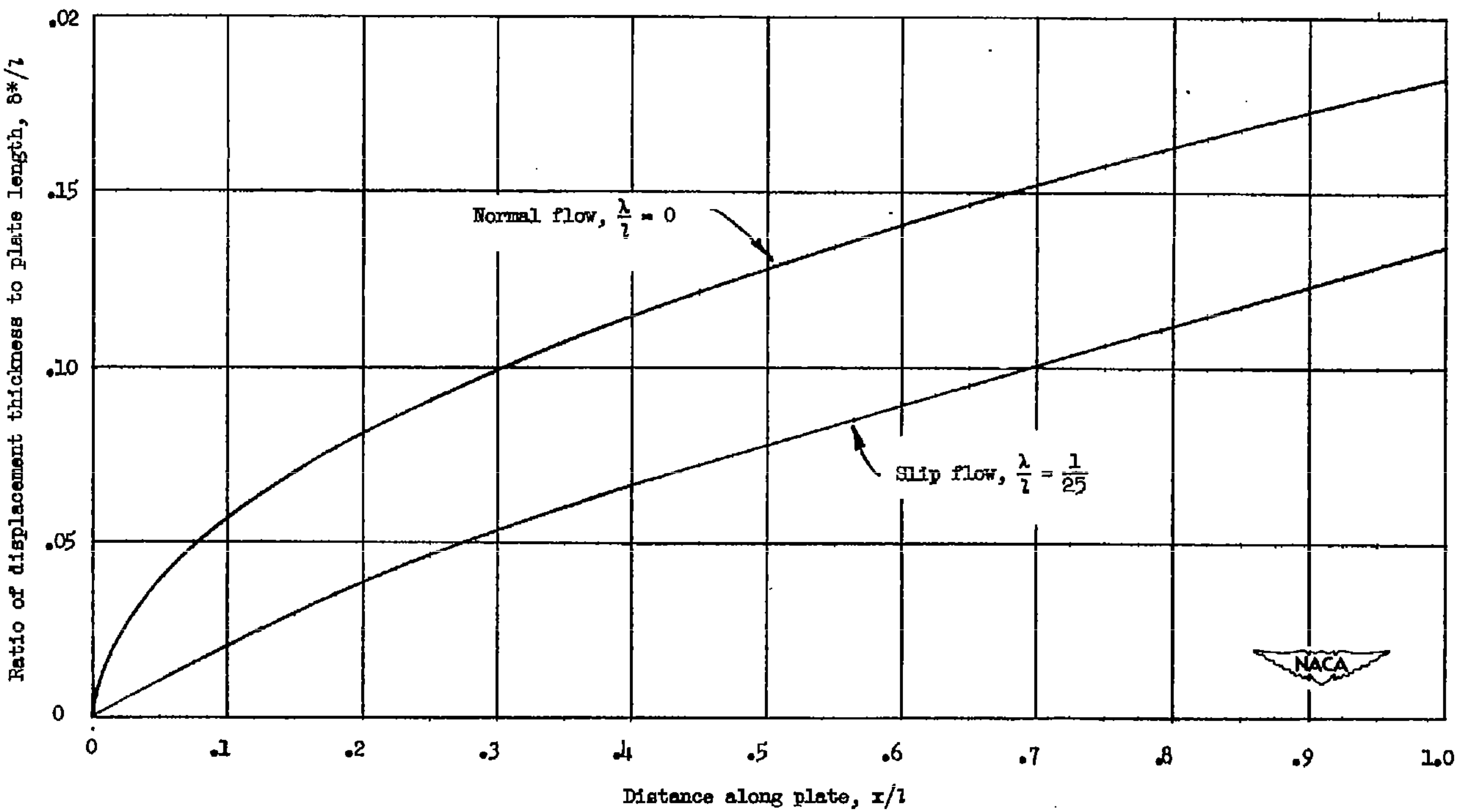

Figure 4.– Comparison of displacement thickness of normal and slip boundary layers. $R_N = 100$.

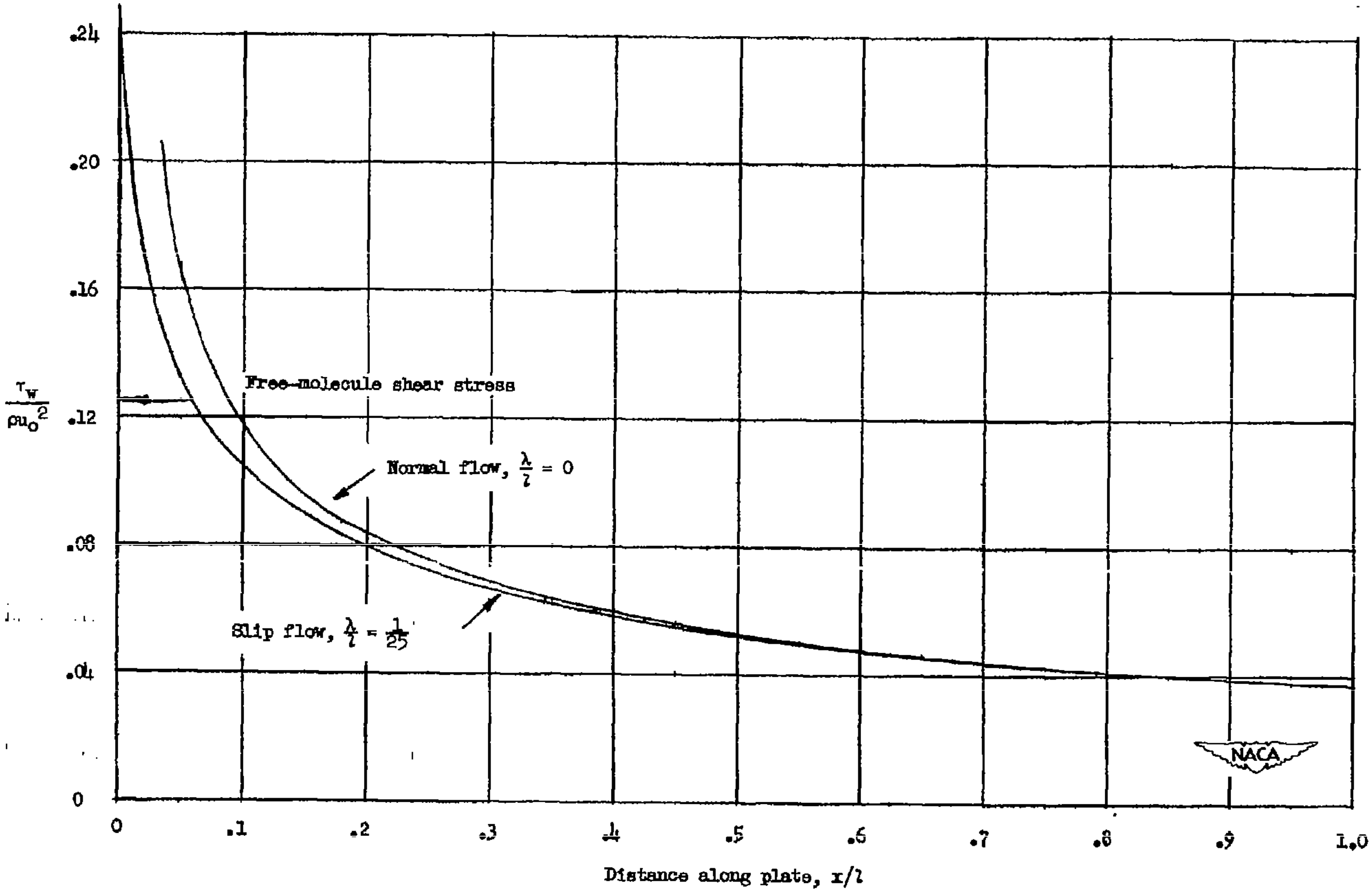

Figure 5.– Comparison of shear stress for normal and slip boundary layers on a flat plate. $R_N = 100$.

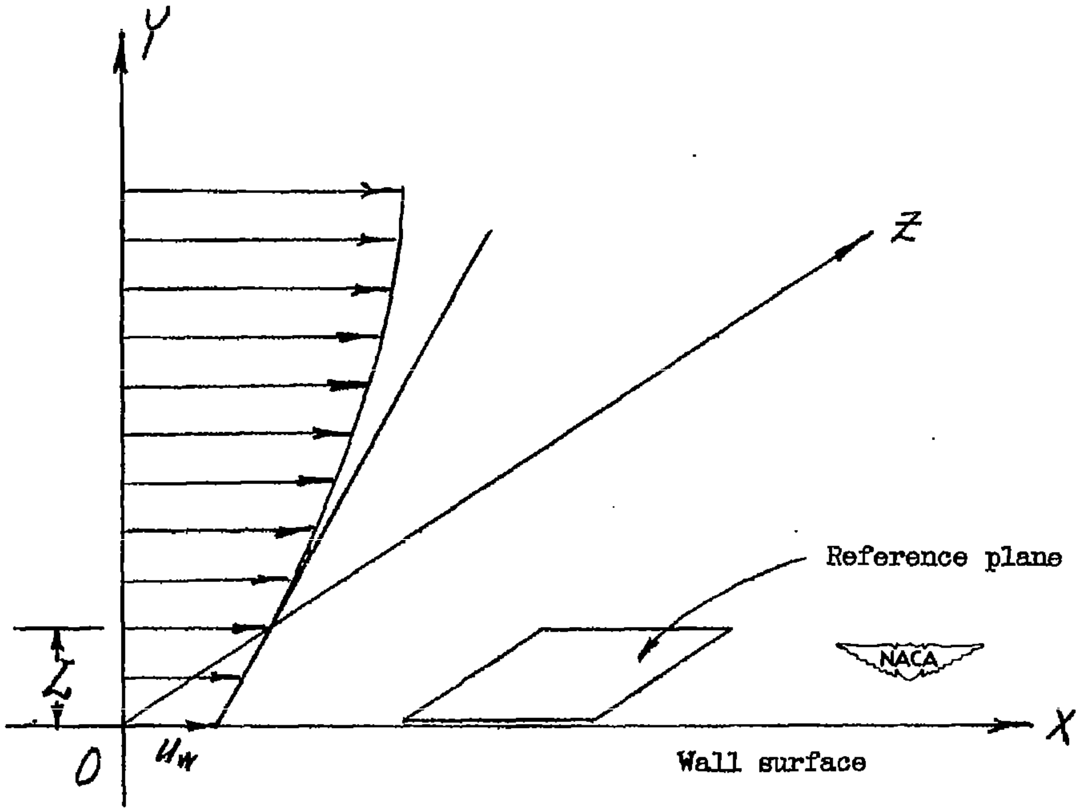

Figure 6.– Coordinate system used to derive momentum transfer equations.